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A note on energy extremal properties for rotating stars in general relativity

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Abstract. Using Komar's localized energy and spin conserved vectors associated with the Killing fields of a stationary axisymmetric universe, we derive a covariant identity which describes in terms of variations of fields and matter functions a perfect fluid in general relativity, exhibiting in a clear and straightforward way a variety of known extremal theorems, or analogous ones, for energy or entropy.

1. Introduction

Thorne and Wheeler (1965) established the following energy extremal theorem: 'Among all momentarily static and spherically symmetric configurations of cold catalysed matter which contain a specified number of baryons, that configuration which extremizes the mass as sensed from outside satisfies the TOV general relativity equations of hydrostatic equilibrium' (Harrison *et al* 1965). Cocke (1965) obtained similar results in terms of an entropy extremal which represents, though it is not evident at first sight, a more precise formulation of the Tolman–Ehrenfest (1930) thermal equilibrium conditions. In any case the 'constraints' (Dirac 1958) or initial value equations are the source of all the information.

Analogous but not completely equivalent properties have been established for a stationary axisymmetric distribution of a perfect fluid by Boyer and Lindquist (1966), Hartle and Sharp (1967) and in a more complete sense by Bardeen (1970). In these variational or extremal principles there is a great difference in the treatment of on the one hand quantities like baryon number, entropy, internal energy and volume (as measured in locally comoving frames) which are all associated with locally conserved vector densities and on the other energy and spin for which, as is well known (Møller 1958), there are troubles of localization in general relativity.

For static or stationary axisymmetric universes in which well defined Killing fields exist everywhere, energy and spin may be to some extent localized; in other words (Komar 1959), conserved energy and spin vector densities also exist. In view of this, we shall show that there exists an identity, relating the variation of the relevant conserved vector densities, which expresses in a direct and covariant way the results derived from a variational principle. We shall first give a very brief description of the mathematical background; results will be followed by some comments on the various applications which have been made.

2. Mathematical treatment

Given a Riemannian manifold in general relativity (Lichnérowicz 1955) with two Killing fields, a timelike one $\xi^\lambda(\lambda, \mu, \nu, \rho, \sigma = 0, 1, 2, 3)$ and a spacelike one η^λ, there exists a family of 'cylindrical coordinates' $(x^\lambda) = (t, \phi, \rho, z)$ in which $\xi^\lambda = \delta^\lambda_0$ and $\eta^\lambda = \delta^\lambda_1$. We suppose that in these frames, the metric $g_{\mu\nu}\,dx^\mu\,dx^\nu$ is invariant for $(t, \phi) \to (-t, -\phi)$, so that $g_{02} = g_{03} = 0$ (the condition may be somewhat weakened). We suppose also that coordinates may be chosen so that $g_{\mu\nu}$ tends to the Minkowski metric at spatial infinity ($r^2 \equiv \rho^2 + z^2 \to \infty$ for fixed t) in the following way (Papapetrou 1948):

$$
\begin{aligned}
\hat{g}^{00} &= 1 + (M/2\pi r) + O(r^{-2}) \\
\hat{g}^{0k} &= +\delta^k_1 J(\rho/4\pi r^3) + O(r^{-3}) \\
\hat{g}^{kl} &= -\delta^{kl} + O(r^{-2}) \qquad (k, l, m, n = 1, 2, 3)
\end{aligned}
\tag{1}
$$

where M and J are the geometrical mass and spin of an isolated perfect fluid (see below); the symbol $\hat{}$ on $g^{\mu\nu}$ (or on any other defined symbol) indicates multiplication by the square root of $-\det(g_{\mu\nu})$.

A perfect fluid is described by a set of geometrical quantities, zero outside an assumed connected finite region, and whose Lie derivatives in the direction of the Killing fields are zero (a symmetry condition). The matter tensor is

$$T^{\mu\nu} = (\sigma + p)u^\mu u^\nu - pg^{\mu\nu} \tag{2}$$

where σ is the internal energy density function, p is the local pressure, and the velocity field u^λ is taken to be normalized, $u^\lambda u_\lambda = 1$. The symmetry condition implies that u^λ is in the plane of ξ^λ and η^λ; thus scalar functions $\bar{\xi}$, Ω exist such that

$$v^\lambda \equiv \bar{\xi} u^\lambda = \xi^\lambda + \Omega \eta^\lambda \tag{3}$$

in cylindrical coordinates, $\bar{\xi} = 1/u^0$ and $\Omega = u^1/u^0 = d\phi/dt$ is the angular velocity. A scalar entropy function s is also assumed to exist and $s = s(\sigma, n)$ where n is the baryon number density function; first-order derivatives define the local temperature θ and the local pressure (equation of state) in the following combination:

$$\theta\, d(s/n) = d(\sigma/n) + p\, d(1/n). \tag{4}$$

Consider now a space-like hypersurface Σ extending to infinity, $d\Sigma_\lambda$ being the local surface element. The Komar (1959) energy and spin forms are defined respectively by

$$\mathscr{M} \equiv \partial_\mu(\nabla^\lambda \hat{\xi}^\mu - \nabla^\mu \hat{\xi}^\lambda) d\Sigma_\lambda \tag{5}$$

$$\mathscr{J} \equiv -\tfrac{1}{2}\partial_\mu(\nabla^\lambda \hat{\eta}^\mu - \nabla^\mu \hat{\eta}^\lambda)\, d\Sigma_\lambda \tag{6}$$

where ∇^λ represents a covariant derivative relative to x^λ, and ∂_μ an ordinary derivation. $\mathscr{M}$ and $\mathscr{J}$ have zero exterior differentials and their integrals are Σ-independent numbers, which because of (1) are equal to M and J respectively.

With u^λ one may define a volume element $\mathscr{V}$ as measured in locally comoving frames, $\mathscr{V} \equiv \hat{u}^\lambda d\Sigma_\lambda$, and with it an internal energy form $\mathscr{E} \equiv \sigma\mathscr{V}$, an entropy form $\mathscr{S} \equiv s\mathscr{V}$ and a baryon number form $\mathscr{N} \equiv n\mathscr{V}$. Their integrals are Σ-independent numbers called, respectively, the volume V, the internal energy E, the entropy S and the baryon number N.

Consider a small change in the metric $\delta g_{\mu\nu}$ and in the independent matter functions $\delta\sigma$, δn and $\delta\Omega$ with the restrictions that the symmetry is preserved $\delta\xi^\lambda = \delta\eta^\lambda = 0$, coordinates are unchanged $\delta x^\lambda = 0$ and u^λ remains normalized. Variations of the forms are easily calculated as are those of the tensor $L^{\mu\nu}$ defined as follows, with the Ricci curvature $R^{\mu\nu}$ and its trace R:

$$L^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R - T^{\mu\nu} \tag{7}$$

($L^{\mu\nu} = 0$ are of course Einstein's equations). Moreover, with a definite value for $\delta\mathscr{V}$, (4), written in terms of forms, becomes

$$\theta\, \delta\mathscr{S} + \mu_0\, \delta\mathscr{N} = \delta\mathscr{E} + p\, \delta\mathscr{V} \tag{8}$$

where $\mu_0 \equiv (\sigma + p - \theta s)/n$ is the chemical potential (rest mass included). Now, $\bar{\xi}$ times the second member of (8) may be expressed in terms of $\delta\mathscr{M}$, $\delta\mathscr{J}$ and $\delta L^{\mu\nu}$ and in fact, using (1) to (7), it may be shown after some simple but lengthy algebraic manipulations that (8) is equivalent to what follows in which $T \equiv \bar{\xi}\theta$ and $\mu \equiv \bar{\xi}\mu_0$:

$$\delta\mathscr{M} - (\Omega\, \delta\mathscr{J} + T\, \delta\mathscr{S} + \mu\, \delta\mathscr{N}) = \mathscr{D} + (v^\nu\, \delta\hat{L}^\lambda_\nu + \tfrac{1}{2} v^\lambda \hat{L}_{\mu\nu} \delta g^{\mu\nu})\, \mathrm{d}\Sigma_\lambda \tag{9}$$

with

$$\mathscr{D} \equiv \tfrac{1}{2}(\delta\mathscr{M} + \hat{g}^{\mu\nu}(\delta R_{\mu\nu}) v^\lambda\, \mathrm{d}\Sigma_\lambda). \tag{10}$$

Interpretation will be easier in cylindrical coordinates and on the hypersurface $t = 0$ for which $\mathrm{d}\Sigma_\lambda \equiv \delta^0_\lambda\, \mathrm{d}^3x$ and $n_\lambda = \delta^0_\lambda (g^{00})^{-1/2} \equiv \delta^0_\lambda \epsilon$. Then, the independent constraints are $\mathscr{H}_L \equiv \hat{L}^{00}\epsilon = 0$ and $\mathscr{H}_\phi \equiv \hat{L}^0_1 = 0$, while the dynamical equations are $\hat{L}_{kl} = 0$, and they are derivable from an Hamiltonian H which is as follows (with $\omega \equiv g^{01}/g^{00}$):

$$H = \int_\Sigma \mathscr{H} \equiv \int_\Sigma \xi^\nu \hat{L}^\lambda_\nu\, \mathrm{d}\Sigma_\lambda = \int_{t=0} \hat{L}^0_0\, \mathrm{d}^3x = \int_{t=0} (\mathscr{H}_L \epsilon - \mathscr{H}_\phi \omega)\, \mathrm{d}^3x. \tag{11}$$

Now defining a form

$$\mathscr{J}^* \equiv \mathscr{J} + \eta^\nu \hat{L}^\lambda_\nu\, \mathrm{d}\Sigma_\lambda \equiv \mathscr{J} + \mathscr{H}_\phi\, \mathrm{d}^3x \tag{12}$$

whose integral J^* is equal to J when $\mathscr{H}_\phi = 0$, and noting that $\mathscr{D}$ is a spatial divergence which because of (1) is asymptotically like $O(r^{-4})$, we may write the integral of both members of (9) as follows, using (11) and (12):

$$\delta M - \int_{t=0} (\Omega\, \delta\mathscr{J}^* + T\, \delta\mathscr{S} + \mu\, \delta\mathscr{N}) = \int_{t=0} (\epsilon\, \delta\mathscr{H}_L - \omega\, \delta\mathscr{H}_\phi + \tfrac{1}{2}\hat{L}_{kl}\, \delta q^{kl})\, \mathrm{d}^3x \tag{13}$$

where $q^{kl} = g^{kl} - n^k n^l$ is the reciprocal matrix of g_{kl}. Formula (9) or (13) is just a variational identity which contains the mentioned extremal theorems or closely analogous ones as we shall now briefly point out. In fact, from (5), (7) and (11) we note that

$$\mathscr{A} \equiv \tfrac{1}{2}\mathscr{M} - \mathscr{H} = \xi^\nu (\hat{T}^\lambda_\nu + \tfrac{1}{2}\delta^\lambda_\nu \hat{R})\, \mathrm{d}\Sigma_\lambda = (\hat{T}^0_0 + \tfrac{1}{2}\hat{R})\, \mathrm{d}^3x. \tag{14}$$

$\mathscr{A}$, up to a divergence, is the integrand of the energy integral obtained from the Noether theorem for standard Lagrangians. Variational considerations have been applied to the integral of $\mathscr{A}$ hereafter denoted A. When applying the variational principle, boundary value variations of the field components are equated to zero; thus (see equation (1)) $\delta M = \delta J = 0$. In these conditions, and using (14), (13) reduces to

$$\delta A = \int_{t=0} (\Omega\, \delta\mathscr{J}^* + T\, \delta\mathscr{S} + \mu\, \delta\mathscr{N} + \tfrac{1}{2}\hat{L}_{\mu\nu}\, \delta g^{\mu\nu}\, \mathrm{d}^3x). \tag{15}$$

Consider now for instance isentropic variations $\delta\mathscr{S} = 0$, with fixed 'spin' ($\delta J^* = 0$) and fixed baryon number ($\delta N = 0$). With these restrictions, the following relation should be satisfied when applying the variational principle to A:

$$\delta A - \bar{\Omega}\,\delta J^* - \bar{\mu}\,\delta N = \delta \int_{\Sigma} (\mathscr{A} - \bar{\Omega}\mathscr{J}^* - \bar{\mu}\mathscr{N}) = 0 \tag{16}$$

where $\bar{\Omega}$ and $\bar{\mu}$ are numbers (Lagrange multipliers). We may ask equivalently that (16) be true with δA replaced by the identical expression given in (15); then

$$\int_{t=0} \{(\Omega - \bar{\Omega})\,\delta\mathscr{J}^* + (\mu - \bar{\mu})\,\delta\mathscr{N} + \tfrac{1}{2}\hat{L}_{\mu\nu}\,\delta g^{\mu\nu}\,\mathrm{d}^3x\} = 0. \tag{17}$$

Our 13 independent variables have 12 independent combinations of variations (since $\delta\mathscr{S} = 0$); in particular $\delta\mathscr{J}^*$, $\delta\mathscr{N}$ and $\delta g^{\mu\nu}$ are independent. From this it follows that for isentropic variations with fixed spin and baryon number, A will be extremum ($\delta A = 0$) if and only if the angular velocity Ω and the chemical potential times ξ are constants and if Einstein's equations $\hat{L}_{\mu\nu} = 0$ are satisfied. Note that under these conditions (see equation (14)) $A = M/2$, and $\delta A = 0$ will represent an energy extremum if the variations of the variables are restricted to satisfy the variational equations $\delta\mathscr{H}_L = \delta\mathscr{H}_\phi = 0$. Formula (9) or (13) appears as a direct expression of such an extremal energy property.

3. Comments

In connection with (9) and (13), we shall now add a few remarks regarding known results. (i) If in (9) $\mathscr{D}$ is replaced by any divergence whose asymptotic behaviour is like $O(r^{-4})$, thus losing the equality sign, the integral form (13), however, remains valid. In this way M, which in (5) is given by the asymptotic value of g_{00}, may be obtained in terms of the asymptotic value of a spatial component of the metric. This establishes the connection with Bardeen's (1970) result. Note, however, that this can only be done at the expense of a covariant formulation. (ii) Thorne (1967) observed that Hartle and Sharp's results are valid for arbitrary variations of the variables. This is true in so far as their results are regarded as coming from an action principle and not as extremizing the mass. However, both are trivially connected as we saw in (14) and the connection is, in fact, implicitly made by Hartle and Sharp. (iii) Cocke's (1964) entropy extremum theorem for static spherically symmetric stars is based on a redundant set of conditions on the variations of the variables. In fact, as can be seen from (9), one can obtain all of his statements without imposing $\delta\hat{L}_1^1 = 0$. (iv) The Tolman–Ehrenfest (1930) thermal equilibrium conditions are incomplete; they are valid for any variations of the variables restricted by $\delta\mathscr{H}_L = \delta\mathscr{H}_\phi = 0$, as can be seen in (13). (v) An action principle may be applied, in the manner of Boyer and Lindquist (1966), to an integral of $\mathscr{M}$ expressed in terms of matter functions as if not only were the constraints satisfied *a priori* but also $\hat{L}_{kl}q^{kl} = 0$. In this way an analogous energy extremal theorem may be formulated if in addition to $\delta\mathscr{H}_L = \delta\mathscr{H}_\phi = 0$ we restrict the variations also to fulfil $\delta(\hat{L}_{kl}q^{kl}) = 0$, a mixture of constraints and dynamical equations.

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